Green’s function for convection diffusion equation with dirichlet boundary condition using separation variable’s method

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**Abstract**. Heat flow affected by the velocity of heat propagation can be expressed in ma-thematical models and is called convection diffusion equation. The exact solution of this equation is differ according to its initial value. In this paper, an exact solution as a function of its initial value was presented in form of Green Function and its convergence to the steady state condition was also discussed. First, the equation was transformed into a homogeneous diffusion equation and the result was obtained using separation variables method for Dirichlet boundary conditions. This result was then inverted so that the solution of the original problem was obtained. From the simulations, it can be concluded that the solution of diffusion convection equation approaches to the steady state condition. Moreover, the existence of the positive convection part makes the equation approaches slower to the steady state condition than the diffusion equation without convection.

Keywords: Green’s function, diffusion equation with convection, Dirichlet boundary condition, separation of variable method.

1. **Introduction**

Heat energy is one of energy that allows the transfer, change, and exchange in a thermal condition between physical systems. Three types of transfer phenomena were diffusion, convection and radiation. Diffusion occurs when there is different temperature among an object and its surrounding and vanish when both has reached the same temperature. Convection occurs when a fluid or heat flow flows through one place to another plane by fluids. This type of heat transfer process is called convection heat transfer [1]. This paper will discuss heat transfer which combine both diffusion and convection.

The case of heat transfer can be formed a mathematical model in the form of partial differential equations (PDE) and often called the convection diffusion equation. Diffusion convection equation describe physical phenomena where particles, energy, or other physical quantities are transferred into the physical system due to two processes: diffusion and convection [2]. The studied of this equation nowadays has been widely applied not only for physics but also in various field for example biology [3]. Therefore, it is important to present the exact solution of this equation.

In this paper, it will be studied about the reduction of the solution of the convection diffusion equation in a case of convection diffusion heat transfer using initial values and *Dirichlet* boundary conditions stated as follows,

|  |  |
| --- | --- |
|  | (1) |

at domain

,

and with initial values

,

where represents temperature at when , is a diffusion coefficient and is temperature propagation rate. In this paper, we will present the green function for homogeneous convection diffusion equation with Dirichlet boundary condition.

The Green's function is a powerful tool in mathematical methods to solving non-homogeneous problem, differential equations, and partial differential equations. It is said to be a powerful tools because this function can solve the solution of first and second order differential equations both homogeneous and non-homogeneous. Green's function are also functions that can affect a boundary condition with infinite domains and slowly converge using the *dirac delta* function [4]. The Green's function with the notation shows the effect of temperature changes on the position of at . With the operator in the equation , and with Dirichlet boundary conditions, a solution of the Green's function will be find in the convection diffusion equation using the variable of separation method.

Some methods of solving differential equations that can produce the Green's function are variable separation methods, parameter variations, and the expansion of Eigen functions. The variable separation method is used in solving the Green's function equation because variable separation is the simplest method of solving a partial differential equation that has the Green's function form.

Research and studies for the convection diffusion equation have been carried out. An example of a study of the problems regarding convection diffusion equations has been presented using homotopy analysis [2], and the finite difference method [5]. This study will examine the solution to decrease convection diffusion equation using the variable separation method. The variable separation method is a method which is to prepare two or more independent variables in solving the differential equation.Convection diffusion equation is a differential equation that has 2 independent variables namely and . The variable separation method is used, when the boundary conditions are used in solving a linear and homogeneous partial differential equation solution [6]. The general solution obtained from the convection diffusion equation using the variable separation method is then an exact solution will be found using initial value problem. From solving the initial value problem, an exact solution is obtained that has a form of Green's function.

1. **The Derivation of Green’s Function**

The general solution will be searched on the equation (1) using the initial problem solving value and boundary conditions with initial values and homogeneous *Dirichlet* boundary conditions

|  |  |
| --- | --- |
|  | (2) |

which will be transformed in the form of an initial value and an upper bound condition of . The Green's function will be sought in the general solution of the convection diffusion equation with form

due to translational invariance.

## Transformation

The initial step in solving the convection diffusion problem is the transformation of the equation form (1) of to

|  |  |
| --- | --- |
|  | (3) |

Then, Equation (3) is derived and substituted into the form of Equation (1).

|  |  |
| --- | --- |
|  | (4) |

Since and are abritatary, taking these value as given in Equation (5) and Equation (6) will transform Eqution (4) into Equation 7 which are known as diffusion equation.

|  |  |
| --- | --- |
|  | (5)  (6) |
| . | (7) |

Next, to get the Green function from Equation (7) is expressed as the multiplication of the function of each independent variable and with the form in the equation

|  |  |
| --- | --- |
|  | (8) |

Derive and substitute the equation (7) to the form (8) and proceed with equating the equation to the independent variables 𝑋 and .

|  |  |
| --- | --- |
|  | (9) |

with as a separation constant.

The results of the separation of the equation (9) form two ordinary differential equations of the variables and which will then be searched for a solution using the boundary condition problem and the characteristic equation.

|  |  |
| --- | --- |
|  | (10)  (11) |

The next step is to find an exact solution of the two ODEs in (10) and (11). In ODE (11), an exact solution will be found using the *Dirichlet* boundary condition as found in Equation (2) of with a value of in each boundary the domain which is

|  |  |
| --- | --- |
|  | (12) |

Because the boundary conditions are used, the ODE (11) is stated as a Boundary Condition Problem which will be solved using the characteristic equation with a boundary condition (12).

* 1. *The General Solution for Diffusion Equation*

A non-trivial solution from boundary condition problem over is obtained by stating is positive, zero and negative at its boundary value as follows.

If is negative, the solution from boundary condition over is

When zero is obtained

so when has a value on will be obtained

Because , then and causes boundary condition problem over does not have a non-trivial solution for with negative value.

If is zero, the solution from boundary condition problem over is

When zero is obtained

so when has a value on will be obtained

Because , there is no non-trivial solution from boundary condition problem for the equation .

If is non-zero positive, the solution of first boundary condition problem is

When zero is obtained

so when has a value on will be obtained

.

Then is chosen to get a non-trivial solution from boundary condition problem over , so

.

So, boundary condition problem over only has a non-trivial solution

Next, the exact solutions will be looked for from ODE's (10). Because the equation (10) is a first order linear ODE with constant coefficient, the solution can be found using the characteristic equation. The equation (10) must meet the boundary conditions of , and . Then, the solution for boundary condition problem for t is,

where is the characteristic equation of Equation (10),

with positive constants and .

In this section, the Green's function will be derived in Equation (1) which transformed from the form to . The first step is to look for which is the general solution of Equation (3). Because the solution is not singular and Equation (3) is a homogeneous differential equations, then can be obtained using the superposition principle.

|  |  |
| --- | --- |
|  | (13) |

* 1. *The Exact Solution for Diffusion Equation*

The next step is to find the value of from the general solution of the equation resulting from the derivating of using the variable separation method and boundary condition problem over and ,

Using the initial value given from Equation (14), the coefficients are obtained based on the orthogonality property of sine.

|  |  |
| --- | --- |
|  | (14) |

Integrating the equation (14) from until we get the value of ,

After getting the value of substitute the value of to the equation by defining the value as a function of .

so the transformation from to is obtained in the form of equation (3) which is

Finally, due to the translational invariance the more general Green's function derived from the convection diffusion equation including elapsed time is,

1. **Simulation**

In this section, the solution of the convection diffusion equation will be simulated using the *wxMaxima* application obtained by the variable separation method, using initial values

|  |  |
| --- | --- |
|  | (15) |

where is the wave amplitude.

Then, determine the value that meets at as the domain limit , as the upper limit of the index , as the diffusion coefficient, and as the convection coefficient. The independent variables entered are , , , , and is bounded by a sigma value with notation . Judging from the previous theory that the solution of the derived Green's function will give the same value as the initial value that was defined. Following are the results of the simulation using *Wxmaxima*,

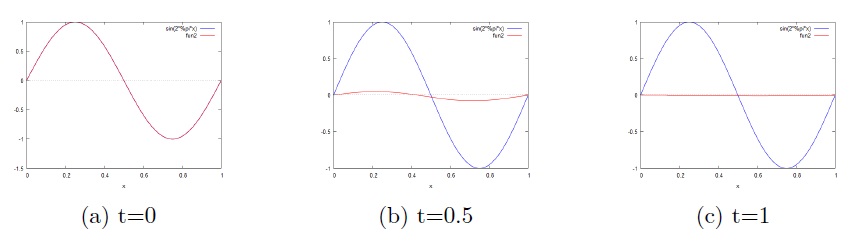


Figure 1. Simulation of Convection Diffusion Equation When

After changing the value from to close to , the graphic will rise close to the initial value of . Next, a plot that moves in the simulation from to , is displayed, after a long time seen in the simulation results of the solution towards a constant condition.

* 1. *Comparison of Simulation with Steady State Convection Diffusion Equation*

Next, a solution of the convection diffusion equation will be sought when it gets the steady state condition which has no effect on temperature so that is considered zero. Then the form of the convection diffusion equation when steady state is

|  |  |
| --- | --- |
|  | (16) |

using *Dirichlet* boundary conditions. Obtained a solution for the equation (16) using the characteristic equation and obtained a solution

|  |  |
| --- | --- |
|  | (17) |

As can be seen in the equation (17), using the boundary conditions is obtained,

Then, using the boundary conditions obtained,

Because , the value of . So, the value of and the solution of the equation (17) is or the solution of the convection diffusion equation when steady state has a trivial solution.

This is consistent with the conclusions in the graphic (Figure 2) simulating the diffusion convection equation that the simulation solution of the convection diffusion equation with an initial value of will eventually lead to the solution of the convection diffusion equation when the condition steady state.

* 1. *Simulation of Homogeneous Diffusion Equations*

Then, the solution of the convection diffusion equation will be compared with the homogeneous diffusion equation by taking the source from [7], using the initial value in the equation (15). The solution of the homogeneous diffusion equation to be compared is

|  |  |
| --- | --- |
|  | (18) |

where is the initial value of the equation solution (18), and is the initial position of .

From the results of substitution of the initial value in the equation (18) using the application , the graphic results for at some point with the value of , , , , , , , , and then compared to the graphic of the solution of the convection diffusion equation, which is

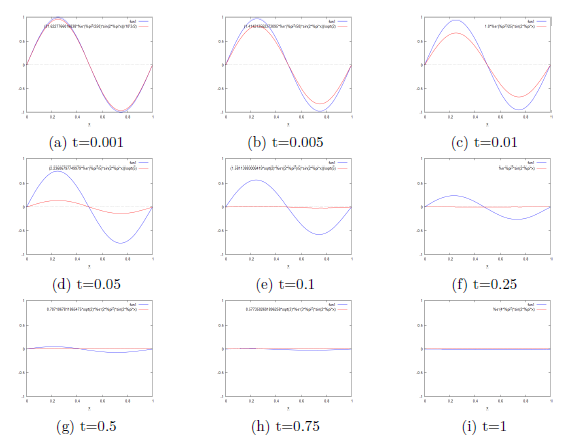


Figure 2 Simulation of Convection Diffusion Equation (Red Line) and Diffusion Equation (Blue Line)

Looking at the picture on (Figure 2) represented by a red line as the motion of the solution of the convection diffusion equation and a blue line as the motion of the solution of a homogeneous diffusion equation, the solution of the equation (18) which has no effect on convection heat transfer can be concluded that the form of the diffusion equation for the case without convection has a faster propagation rate so it is faster to the steady state condition.

1. **Conclusion**

From the discussion, the Green's function with the value of

is obtained from an exact solution of the convection diffusion equation using the variable separation method, which is the result of the transformation from the general solution to .

With the results of the solution of the convection diffusion equation over , simulations are performed using initial values

where is the wave amplitude, and an exact solution is shown with as the initial value of the exact solution of the convection diffusion equation.

So it can be concluded that the exact solution of convection diffusion equation is slowly going to a constant condition that is the solution of the convection diffusion equation when steady state or .

Furthermore, for comparison of solutions to the diffusion equation when there is no convection effect it can be concluded that the solution in the diffusion equation is faster toward the steady state.

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