European Put Option Model Up-and-Out Constant Barrier and Exponential Barrier with Fuzzy Parameters

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**Abstract.** This paper considers an up-and-out European barrier put option model with a constant barrier type and an exponential barrier function type in an uncertain environment. Therefore, the main objective of this paper is to apply a fuzzy approach to analyze the determination of these options with fuzzy parameters. For a set of parameters given by using fuzzy triangular numbers, namely fuzzy interest rate, fuzzy volatility level, and fuzzy stock price, a put up-and-out option model with constant barrier and exponential barrier function is derived in this paper. This paper provides numerical simulations for several explicit parameters and degrees of confidence.

**Keywords**: European put option, up-and-out, constant barrier, exponential function barrier, triangular fuzzy number.

# INTRODUCTION

Today, derivatives are the primary financial instrument for financial markets and trading. Based on Bank for International Settlements (BIS) derivatives statistics, the notional amounts outstanding from the global over-the-counter (OTC) derivatives market traded on all contracts exceeded 598.416 trillion, with a Gross market value of 12.439 trillion in Semester 2 2021 (<https://stats.bis.org/statx/srs/table/d5.1>). With the large trading volume and gross market value in the derivatives market, there is an increasing market share in options trading. Therefore, determining the value of options becomes theoretically and practically essential for traders and investors.

Black and Scholes first obtained a definitive solution for determining the value of European options on stocks in 1973[1]. Due to fluctuations in financial markets over time, some of the input parameters in the Black Scholes model cannot always be determined in a clear and accurate sense [2]. Therefore, a study of the application of the fuzzy set theory was developed, which was introduced by Zadeh in 1965 to describe inappropriate parameters in financial mathematical models. Using fuzzy parameters in the mathematical model is expected to provide a reasonable range and appropriate membership for option values. Thus, investors may interpret optimal values differently for different risk preferences.

In the last decade, to describe imprecise parameters in financial mathematical models, a study of the application of fuzzy set theory introduced by Zadeh in 1965 was developed, which was used in determining the value of European options, American options, and real options. The first research related to the determination of option value in a fuzzy environment was introduced by Buckley in 1987. Buckley developed a fuzzy pattern of future value, present value, and internal rate of return (IRR) [3]. Yoshida et al. consider fuzzy numbers and fuzzy stochastic processes for determining the value of American put options [4]. A review of the literature that discusses direct and indirect problems in determining option values ​​in a fuzzy environment is carried out by Muzzioli and De Baets [5]*.* In 2018, Jorge de Andre´s-Sanchez evaluated the Black Scholes model for determining the value of European options with fuzzy triangular numbers. Empirical applications were carried out using a sample of options on IBEX35 traded in the Spanish derivatives market [6]. In the same year, Mezei presented an extension of the fuzzy payoff method for determining the value of real options using interval-valued triangular fuzzy numbers [7]. Still, in the same year, Xu et al. derived the option value determination model with a fuzzy binomial tree approach to model firm value and recovery rate as fuzzy numbers [8].

Carlsson & Fullér introduced a heuristic determination of the value of real options using a fuzzy approach, where trapezoidal fuzzy numbers estimate the present value of the expected cash flow and expected costs. The optimal training time is determined with the help of the possible mean and variance of fuzzy numbers. Carlsson & Fullér solves the problem of determining the optimal exercise time of real options to delay investment, using the real option fuzzy model, which was developed in 2000, along with the possibilistic mean and variance of fuzzy numbers for approximating the expected value of cash flows and their volatility in future [9]. Xu et al. developed a jump-diffusion model for determining the value of European options under a fuzzy environment [10]. Wang et al. examine the use of fuzzy set theory in Geske's compound option model [11]. Zhang et al. investigated the issue of Asian option value when the underlying asset's interest rate, volatility, and price are represented by trapezoidal fuzzy numbers [12]. Wang et al. examine the Black Scholes model with fuzzy number coefficients[13]. Cheng et al. discuss the Black Scholes model, assuming that stock returns are in the form of Gaussian fuzzy numbers along with Greeks parameter studies [14]. Zhang et al. developed a fuzzy mixed fractional Brownian Motion model with jumps to determine European options' value [15].

This paper develops a fuzzy model of an up-and-out European put option with a constant barrier of an exponential function, assuming that stock prices, volatility levels, and interest rates are fuzzy triangular numbers. Overall this paper is structured as follows. Section 2 introduces the idea and arithmetic of fuzzy numbers. Section 3 presents a fuzzy model of put up-and-out options for the constant and exponential function barriers, assuming that stock prices, volatility levels, and interest rates are fuzzy triangular numbers. The discussion section and the simulation results of the model are presented in section 4, while section 5 deals with conclusions.

# Preliminaries

This section revisits some basic concepts and essential properties associated with fuzzy numbers in Wu's articles [2], [16] and Zadeh's articles [17]. The fuzzy set in , where is the set of real numbers, is the set of ordered pairs , where is the membership function of which maps into the real number interval . Let represent the universal set of all real numbers, then the fuzzy subset is defined by its membership function . The set is defined by . , set from is defined by “*the closure*” of the set , and is called a normal fuzzy set if there is x such that meet these conditions.

**Definition 1.** Fuzzy set in is called a convex fuzzy se, if and only if for any and ,

 (1)

**Definition 2**. The following conditions must be met for A to be defined as a fuzzy number:

(1) is a normal and convex fuzzy set;

(2) The membership function is upper semi-continuous;

(3) The -level set of is constrained to all .

Zadeh proves that if is a fuzzy number, then is a convex and compact set [17]. That is, is a closed interval, denoted by , and has the following properties:

 (2)

Fuzzy sets and membership functions are developments of classical sets and characteristic functions. A number of chips is introduced according to the provisions.

**Proposition 1** (Resolution Identity [17]). Suppose is a fuzzy set with a membership function and set , yaitu , then

, (3)

where is the indicator function of the set , that is, if and if . Note that set is a set of crisp numbers

**Definition 3**. is called a crisp number with the value m if its membership function is

 (4)

then is a crips number denoted by . It easily follows that and any real number can be considered a crips number.

**Proposition 2** **(Extension Principle)**

All fuzzy subsets of are denoted as . Let be a real-valued function from to , and are n-fuzzy subsets in , we get the fuzzy value function from the real value function . In other words, is a fuzzy subset of with the degree of membership:

**Proposition 3** [2]

Let be real function from to and are n fuzzy subsets in . Based the on Extension Principle in Proportion , we get the fuzzy value function from the real value function . Consider every is the upper semi-continuous in for and is a set compact in (closed and finite set in ) for r to be at , then from is

.

Let and are two fuzzy number, with and , then is also a fuzzy number with the set in format:

 (6)

 (7)

 (8)

For all .

If is set without zero, then also fuzzy number with set as follows:

 (9)

**Definition 4**. If the membership function has the following form:

 (10)

then fuzzy numbers is “triangular fuzzy number”, which has a core , with left width , and with right width . Let express triangular fuzzy number, then set has the following form:

 (11)

# UP-AND-OUT EUROPEAN PUT OPTION FUZZY MODEL

Barrier options are a type of path-dependent option, where the value of the barrier option depends on the underlying asset's price during the option period. More specifically, the payoff of the barrier option depends not only on the cost of the underlying asset at maturity but also on whether the underlying asset reaches a barrier value limit during the validity period of the option. In the barrier option, there are knock-in and knock-out features. Barrier knock-in options are active when the underlying asset price reaches a barrier value, and vice versa for barrier knock-out options. If the barrier value is higher than the underlying asset price at t=0, then the barrier option is of the up type.

In contrast, if the barrier value is lower than the underlying asset price at t=0, then the barrier option is of the down type. Thus, there are 8 types of barrier options: up-and-out call options, up-and-in call options, down-and-out call options, down-and-in call options, put up-and-out options, up-and-in put options, down-and-out put options, and down-and-in put options. This paper discusses the development of European barrier put options, with the up-and-out type being used. An up-and-out put option is a European put option that becomes inactive as soon as the stock price reaches or crosses barrier B before expiration. Initially, the cost of the S stock is lower than the barrier value of B and must remain low or fall for the option to stay active.

## Constant Barrier Option Fuzzy Model

In this section, we discuss the development of a put up-and-out option model by assuming a constant barrier which is written as follows.

 (16)

with condition for case and condition for case .

Suppose is the value of the European put option up-and-out barrier constant over time . The formula for the value of the put option is a constant up-and-out barrier at the stock price , with maturity times , strike price , barrier , volatility , and interest rate under the assumption that the strike price is lower from the barrier value and no rebate payment, expressed as follows[18].

 (17)

where

, (18)

 (19)

, (20)

 (21)

 (22)

and is the cumulative distribution function of the standard normal random variable .

Based on the consideration of uncertainty conditions in the parameters of stock prices, interest rates, and levels of volatility, these parameters are expressed as non-negative fuzzy parameters. Because of the strike price , maturity time , and barrier are real numbers, then in this model, the three parameters are expressed in crips numbers, namely , , and with value , , dan . The fuzzy value of the put option up-and-out constant barrier at time t is denoted by where , and is a fuzzy random variable, for all . The fuzzy put up-and-out constant barrier model with fuzzy interest rate parameters , fuzzy volatility and fuzzy stock price given with

 (23)

where

 (24)

 (25)

 (26)

 (27)

 (28)

According to “Resolution identity” in Proposition 1, the membership function is given by:

 (29)

where is set of and can be written in a closed interval, as follows:

 (30)

Based on Proposition 2, since the cumulative distribution function is an increasing monotone function, then set is given by:

As well as, because is a decreasing monotone function, dan is an increasing monotone function, then set , , are

Then, using proposition 2 and equations (6), (7), (8), and (9), is the left endpoint and is the right endpoint of the closed interval is determined by the following formula:

 (31)

 (32)

where

 (33)

 (34)

 (35)

 (36)

 (37)

 (38)

 (39)

 (40)

 (41)

 (42)

## Exponential Function Barrier Option Fuzzy Model

Moving curve barrier is a barrier in the form of a curve. This put up-and-out option model assumes that the barrier is an exponential function which is written as follows

 (43)

with condition for case and for case . The choice of the form of this exponential function is carried out with the hope that the barrier will have the same characteristics as the stock price function, which is a function of increasing over time.

Let be the value of the put up-and-out barrier exponential function at time . The formula for put option value up-and-out barrier exponential function on stock price , with maturity time , strike price , barrier , volatility , and interest rate under the assumption that the strike price is lower from the barrier value and no rebate payment [19], it is stated as follows:

 (44)

where

, (45)

 (46)

 (47)

and is the cumulative distribution function of the standard normal random variable .

The value of the fuzzy put up-and-out barrier exponential function at time t is denoted by where , and are fuzzy random variables, for all . The fuzzy put up-and-out barrier function exponential model with fuzzy interest rate , fuzzy volatility and fuzzy stock price is given by

 (48)

where

 (49)

 (50)

 (51)

Then, using proposition 2 and equations (6), (7), (8), and (9), is the left endpoint and is the right endpoint of the closed interval is determined by the following formula:

 (52)

 (53)

where

 (54)

 (55)

 (56)

 (57)

 (58)

 (59)

# 4. NUMERICAL SIMULATION

In this section, numerical simulations are carried out to determine the value of the fuzzy put up-and-out barrier constant and the exponential barrier function. Consider a European option based on a stock with a strike price of $1650, the expiration time taken for this numerical example is 3 months with 12 months a year, with implied volatility being 0.4048 and interest rate being 0.095, for a constant barrier taken and for exponential barrier taken . Based on these inputs, the value of the constant barrier option is 72.7478, and the value of the exponential function barrier option is 72.2722. This numerical simulation assumes that the volatility is a triangular fuzzy number (0.4000; 0.4048; 0.4900) and the stock price is a triangular fuzzy number (1500; 1560; 1620). All numerical results are designed on the MATLAB 2014b platform and performed on computers with Windows 7 (64-bit), 8G memory, Intel Core i5-8th Gen**.**

Table 1 shows the closed interval values ​​of the European put option fuzzy up-and-out constant barrier and exponential barrier for different degrees of confidence. When the degree of confidence increases, it can be seen that the value of the left test point increases while the right endpoint decreases; this holds for both European put option models, the up-and-out barrier. The higher the confidence interval, the shorter the length of the interval from the closed interval the fuzzy option value

|  |
| --- |
|  **TABLE 1** European up-and-out put option fuzzy value |
|  | **Constant Barrier** | **Exponential Function Barrier** |
| 0.001 | [-423,9240 ; 587,5638] | [-394,9755 ; 543,4426] |
| 0.25 | [-301,8751 ; 457,5510] | [-277,2542 ; 423,8045] |
| 0.35 | [-252,5231 ; 405,6525] | [-230,3095 ; 376,3109] |
| 0.50 | [-178,1428 ; 328,1440] | [-160,1729 ; 305,5427] |
| 0.75 | [-53,2571 ; 199,8723] | [-43,8076 ; 188,5473] |
| 0.85 | [-2.9878 ; 148.8839] | [2,6334 ; 141,9799] |
| 0.90 | [22,2134 ; 123,4588] | [25,8456 ; 118.7289] |
| 0.95 | [47,4586 ; 98,0801] | [49,0573 ; 95,4943] |
| 0.97 | [57,5690 ; 87,9416] | [58,3427 ; 86,2042] |
| 0.99 | [67,6864 ; 77,8105] | [67,6288 ; 76,9158] |

Based on Table 1, suppose for = 0.95, this means that the value of the up-and-out European put option will be in the closed interval [47,4586 ; 98,0801] with a confidence level of 0.95. From another point of view, if a financial analyst is comfortable with this 0.95 confidence level, they can choose any value from the closed interval [47,4586 ; 98.0801] as an option value for later use..

# CONCLUSION

This paper analyzes the determination of the fuzzy option barrier value with the European put type for the up-and-out type with a constant barrier and an exponential barrier function in an environment of uncertainty. The analysis was carried out by applying the use of fuzzy parameters. This paper presents stock prices, volatility, and interest rates in fuzzy triangular numbers to uncertain model parameters. Financial analysts give several different degrees of confidence, and overall for the value of the option for both the constant barrier and the exponential barrier, increasing the degree of confidence decreases the length of the closed interval for the value of the put up-and-out option. The interval length of the put up-and-out barrier value is exponentially shorter than the interval length of the same option value with a constant barrier because the exponential function barrier is an ascending function. Furthermore, fuzzy parameters can be used to develop models for other types of barrier options.

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