Undergraduate students’ levels of understanding in solving mathematical proof problem: the use of Pirie-Kieren theory

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**Abstract**. One of problem solving cases that is often encountered by undergraduate students of mathematics education study program in various courses is the mathematical proof problem. This problem is encountered starting from the first year to the last year. This research used a qualitative approach that aimed to describe the levels of students’ understanding in solving mathematical proof problem. The description used Pirie-Kieren's theory. In Pirie-Kieren's theory, there are 8 levels of understanding, namely primitive knowing, image making, image having, property noticing, formalizing, organising, structuring, and inventising. The subjects in this reserach were students of Mathematics Education study program of Universitas Sulawesi Barat who were taking Abstract Algebra course in the academic year of 2019-2020. Subjects consisted of 6 students with three levels of abilities: high, medium, and low. Each category was represented by 2 students. Data were obtained using mathematical proof problem test and interview guidelines. Based on the triangulation of data obtained that the highest level of understanding in solving mathematical proof problem achieved by the high category subjects is structuring, the highest level of the medium category subjects is organising, and the highest level of the low category subject is property noticing.

1. Introduction

Logical reasons are indispensable in discussing any field. When discussing law matters, people will use the constitutional law articles to support their logical reasons. When talking about history, historical evidence or documents are mandatory as a basis for discussion. Logical reasons will not be accepted without proof as a form of validation of the arguments put forward. Likewise in mathematics, proof is used to validate mathematical statements.

Mathematics is based on valid relationships and laws and characterized by deductive knowledge construction. Mathematics comes from individual opinions. Therefore, reasoning, attitudes and arguments are needed as an integral part of everyday life. Mathematical thinking, reasoning and the use of arguments are skills needed to construct mathematical proof as one of the high-level skills that indicate the level of mathematical literacy. For in-depth understanding of mathematical principles and concepts, mathematical proof is a necessity [1]. Because it is the basis of mathematics, students are required to learn, understand concepts and construct proofs [2]. Some experts also expressed the importance of proof in mathematics [3], [4], [5], [6].

However, how is the existence of mathematical proof in the world of education itself? Some research results show that at the secondary to undergraduate level, it is still difficult for students to prove mathematically even though they are at undergraduate level or prospective teachers [7]. Mathematical proof is a difficult work. In fact, mathematical proof is a must-have ability because it is essential for learning mathematics at the next level [8]. Proofing can be an excellent activity to do in the classroom to develop their mathematical understanding. But what often happens in class, mathematical proof is perceived by students as something that must be learned by memorizing. This method only serves to re-emphasize that mathematics is about learning facts and procedures by heart, whereas the purpose of the concept of proof is often not explained [9]. Students still have difficulty constructing what arguments they use in proving, understanding the methods of proof, even to differentiate between what are proof and what are not [10]. Leveling the understanding of students' in doing mathematical proof will be very useful to see the extent of their ability to prove, including what difficulties they have faced and what efforts are made when faced with problems of mathematical proof. One of the theories of understanding used in the world of mathematics is the Pirie-Kieren theory.

## Mathematical Proof

In the discipline of mathematics, mathematical proof is very central [11]. Proof in mathematics is a comprehensive argument, to show that certain statements are assumed to be true. Proof can be thought of as a series of logical arguments and claims that clearly show that the statement is true. Often times, we construct proofs using theorems, statements that have previously been proven true, and axioms, statements that are universally accepted without the use of proof [12]. “Proof is a formal demonstration of a result, a sequence of logical arguments that allows establishing the veracity of a mathematical property. Proof is assumed to be central in mathematics” [13].

At the undergraduate level, mathematical proof is a fundamental component of the curriculum of the mathematics department. So important is mathematical proof, some universities even made a special transition to teach proof. The transition includes basic logic and various methods of proof in mathematics in various areas of mathematics such as set theory, number theory, discrete mathematics, abstract algebra, and real analysis [13], [14], [15].

There are several important aspects that can be given to students in identifying and developing mathematical proof abilities, namely (a) mathematical proof is not built on empirical proof, (b) uses mathematical properties, (c) effective argumentation techniques, and (d) construct what is already known [9].

## Pirie-Kieren Theory

What is meant by understanding in mathematics? The term understanding is still very broad to be interpreted, especially in the world of education. The term "understanding" can be narrowed down for example understanding the definition of mathematics, theorem or proof, theory, method, algorithm, problem, or solution [16].

There are several theories about understanding used in mathematics education. One of them was proposed by Pirie-Kieren [17]. The Pirie-Kieren theory of understanding provides a framework for analyzing a person's understanding with inconsistent movement from one level to another [18]. The Pirie-Kieren theory is recursive. The process is dynamic, not linear. There are 8 levels of understanding embedded from early primitive knowing to inventising [19]. These levels of understanding are illustrated by eight layers of rings. Each layer represents a different level. We can list the levels of understanding as follows: primitive knowing, image making, image having, property noticing, formalising, organising, structuring, and inventising [17].



**Figure 1.** The Pirie-Kieren Model for Levels of Understanding

If it is related to mathematical proof problems, the indicators for each level of understanding based on the Pirie-Kieren theory can be made as in Table 1 [20].

**Table 1.** Indicators of Mathematical Understanding Based On Pirie-Kieren Theory

| No | Level Level | Indicators |
| --- | --- | --- |
| 1 | *Primitive knowing* | * Understand all definitions of the terms found in the problem
 |
| 2 | *Image making* | * Get ideas or decriptions that will be used in solving problem
* Explain the idea or description of the solution through examples
 |
| 3 | *Image having* | * Get ideas or decriptions that will be used in solving problem
* Explain the idea or description of the solution without examples
 |
| 4 | *Property noticing* | * Realizing the relationship between definitions understood in the primitive knowing stage
* Verify the relationship between these definitions
 |
| 5 | *Formalising* | * Creating a concept related to the definition of the relationship
 |
| 6 | *Organising* | * Uses the concepts found to solve the given problem
* Finding structured patterns from concepts to solve a given problem
* Make a formal statement from the patterns found to solve the given problem
 |
| 7 | *Structuring* | * Connceting the relationship between one theorem and another and able to prove it based on logical arguments
 |
| 8 | *Inventising* | * Have a complete structured understanding and create new questions that can grow into a new concept
 |

1. Methodology

This research is a research with a qualitative approach that aims to describe the level of understanding of students in solving mathematical proof problems.

## Participants

Initial data in this research were obtained from 67 students from 2 classes of the Mathematics Education Study Program of Universitas Sulawesi Barat for the 2019/2020 academic year. At the time of data collection, the student was programming the Abstract Algebra course. Of the 67 students, 6 students were selected to be the subjects of this research.

## Data Collection Procedures

Participants in this study were first given a mathematical proof test consisting of 1 question to prove the Group Theory. Furthermore, from 67 students, 6 students were selected based on their mathematical proof ability category, namely high, medium, and low, and 2 students were selected for each category. The six students will be conducted in-depth interviews to see their level of understanding in solving mathematical proof problems

* 1. *Data and Data Analysis*

There are two types of data obtained in this study: (1) data from the results of mathematical proof tests which are used to categorize prospective research subjects as well as sources of interview material and (2) data from interviews using semi-structured interview guidelines. The two data collection methods are method triangulation as a form of data validation in this study. In addition, source triangulation is also used by collecting data from 2 subjects in each category. From the triangulation results, conclusions will be drawn.

1. Result and Discussion
	1. *Result*

Giving a mathematical proof test as a first step in determining the subject in this study resulted in the data presented in the following figure.

**Figure 2.** Mathematical Proof Test Scores

From the data obtained above, 2 subjects were selected for each category and obtained the following data.

Table 2. High Category Subjects Result

|  |  |
| --- | --- |
| Subject | Interview Summary |
| H1 | * The subject was able to describe and explain well what the problem would prove
* The subject used the elements given in the questions to start the proving process and then broke them down into new elements to be used in the proof
* The subject made use of previously known properties and theorems in the process of proving
* The proofing process was carried out in a very systematic and structured manner
* The subject was able to draw final conclusions that were correct and very much in accordance with what the question wanted to prove
 |
| H2 | * The first procedure performed by the subject in solving the problem of proof was to use the known elements in the problem to bring up new elements that would be used to prove
* The subject was able to explain what was being shown in this proof
* The subject made use of the known elements of the problem and their properties and theorems in the process of proof
* The subject did not write conclusions on the answer sheet but was able to explain the final process of proof very well
 |

Table 3. Medium Category Subjects Result

|  |  |
| --- | --- |
| Subject | Interview Summary |
| M1 | * The subject was able to explain the meaning of the problem shown by decomposing the problem into a mathematical statement to be shown
* Furthermore, from the known elements in the problem, the subject obtained new equality which would be used for the next phase
* The subject raised a theorem which according to them was related to the proving process and tried to use the theorem to get what he wanted to show
* The subject was actually able to understand what was being shown but it was difficult to write and explain the steps that were systematic and easy to understand
* The subject wrote conclusions but were unable to clearly explain the process of obtaining them
 |
| M2 | * The subject raised new equality obtained from the known elements of the problem
* The subject was able to explain what similarities are being shown as a process of proving the problem
* The subject worked the proof by using the method for two
* The subject had difficulty in explaining the two methods, some of the steps described were wrong and confused
* The subject wrote a conclusion but had difficulty explaining how to obtain it from the proof method used
 |

Table 4. Low Category Subjects Result

|  |  |
| --- | --- |
| Subject | Interview Summary |
| L1 | * The subject tried to explain the initial steps of proof but wAS only able to mention what elements were in the problem
* The subject tried to remember the material related to the problem but wrote and explained it was not quite right
* The subject broke down the process of proof using the method of dividing by two and led to an equality but had no relation to what was being shown
* The subject made a conclusion but it had nothing to do with the proofing process described earlier
 |
| L2 | * The subject tried to explain the initial steps of proof but was only able to mention what elements were in the problem
* The subject used the known elements in the problem to start the proof, but the direction of the proof was still not clear
* The subject tried to make a conclusion according to the final steps of the proofing process carried out but it was not an answer to the problem
 |

* 1. *Discussion*

Referring to the indicators of Pirie-Kieren's theory of understanding and the results of the research in the previous section, the levels of understanding achieved by the high category subjects are **primitive knowing**. This can be seen from the ability of the two subjects to explain the meaning of all known elements in the problem. The next level that is achieved is **image having**, both subjects are able to write and explain exactly what they want to show from the given proof problem. The subject explains the idea without giving examples, or in other words the subjects find the idea in general. The third level achieved by the two subjects in this high category is **property noticing**. This is shown by the subjects’ ability to use these elements correctly to bring up new elements that will be used in their proofs, the relationship described is also very precise. Both subjects were also able to reach the formalizing level. From the results of carrying out tests and interviews, subjects are able to come up with the basic concepts of what they will use in proofing. Subjects explain will use the elements in the problem, the new elements they get, and the various definitions and theorems they have obtained previously. When they did the evidence, it could be concluded that the two subjects were also able to reach the **organising** and **structuring** levels. Subjects are able to explain the proof process using the concepts they designed at the previous level, work in a very structured and systematic manner, and use definitions and theorems appropriately.

The level of understanding that can be achieved by the medium category subjects is **primitive knowing**. Even though they do not show it explicitly in their work, the subjects are able to explain exactly the meaning of the problem which indicates that the subjects understand all the terms that appear on the problem. Similar to the high category subjects, these two medium category subjects are able to explain their proof plan by explaining what equality will be shown so that they can prove the problem, without going through examples, so that at this stage, the subject is able to reach the level of **image having**. The next level to be achieved is **property noticing**. The subjects realize that the elements that exist in the problem will be used in the proving process, so the subjects use these elements to bring up new elements. After that, the subjects use these new elements to create a concept or proof plan, at this stage the subject is at the level of **formalizing**. At the next level, namely the **organising** level, the subject is able to use the concept at the formalizing level and successfully finds patterns and tries to complete the process of proof using formal mathematical arguments, however, both subjects still have difficulty using existing theorems to get to the final result.

The low category subjects were also able to understand the terms in the questions. At the time of the interview the subjects were able to explain the meaning of all the written definitions, both subjects were at the **primitive knowing** level. Unlike the high and medium category subjects, the low category subjects tried to present their ideas in evidence but not in general terms, but tried to remember related examples that they had previously received. Here, the subjects are at the **image making** level. Furthermore, the two subjects used their ideas to start the proofing process, the subjects also used the definitions of the problem. However, the subjects have not been able to explain the correct concept for them to use in the proofing process. The highest level achieved by the low category subjects only reached the **property noticing** level.

In accordance with the basic principles of Pirie-Kieren theory, the level of understanding is inconsistent, recursive, dynamic, not linear [18], [19]. The high category subjects are not at the image making level, but are directly in the image having category, as well as the medium category subjects. Meanwhile, the low category subjects are in the level of image making, unable to reach the level of image having but still able to achieve a higher level of image having, namely the property noticing level.

1. Conclusion

The results of the research and discussion in this research indicate that the level of understanding of the high category subjects in solving mathematical proof problem is primitive knowing, image having, property noticing, formalising, organising, and structuring. These high category subjects did not go through the image making level. The level of understanding of the medium category subjects in solving mathematical proof problem, namely primitive knowing, image having, property noticing, formalising, and organising. As with high category subjects, medium category subjects also did not go through the image making level. Meanwhile, the low category subjects are only at the primitive knowing, image making, and property noticing levels.

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