Some fixed point theorems in generalized modular metric space

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**Abstract**. In this paper, we discuss some properties of generalized modular metric spaces. This space was first introduced by Turkoglu and Manav in 2018. Later, we extend the concepts about contraction mappings in generalized modular metric spaces. We also develop a convexity structure and defined the normal structure in this space.The results show us that if is a complete modular metric space and is a contraction mapping then has a unique fixed point. Moreover, we need the constant of generalized modular metric axiom should be less than or equal to 1 to guarantee the ball is closed. The closeness property of the ball is necessary to generate normal structure in generalized modular metric space. This normal structure property assures the existence of the fixed point of a nonexpansive mapping.

1. **Introduction**

The fixed point theorem has been developed in 1922 since the appearance of the Banach Contraction Principles. This field is quite popular among mathematicians to develop into various spaces and various types of mapping because of its many applications, for example, to analyze the solution of differential equation systems.

In 2015, Jleli and Samet formed a new space called the generalized metric space. This space already contains metric space, b-metric space, dislocation metric space, and modular space which has Fatou property, but does not contain modular metric space yet [2], [4], [7], [10], [13]. Therefore, Turkoglu and Manav form a spatial structure based on the concept of Jleli and Samet which is called a generalized modular metric space and give some fixed point theorems [7], [14]. Some other results of fixed point theorem in metric space, normed space, modular space, metric modular space, generalized metric space, and generalized modular metric space also can be seen in [1]-[14].

1. **Preliminaries**

First, let’s recall the definition of generalized modular metric [14].

*Definition 2.1.* Let be an arbitrary nonempty set and is a function on . The function is called generalized modular metric space if it satisfies the following conditions:

(A1) If for some then for all .

(A2) For every satisfies for all .

(A3) There is exist such that for all and { which for some then .

The pair is called generalized modular metric space. For convenience, we write instead of .

The connection between generalized modular metric space, metric space [4], and modular metric space [2] can be seen in Figure 1. Moreover, generalized modular metric space also contains the generalized metric space [7].

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| Modular metric space  Generalized modular metric space  Metric space |
| **Figure 1**. The connection between metric space, modular metric space, and generalized modular metric space. |

Some notions have been introduced in generalized modular metric space [14], [6]. But instead of we define them in , we can define them in .

*Definition 2.2* Let be a modular metric space and .

1. Any sequence is said -convergent to if only if for . Moreover, is called -limits of .
2. Any sequence is called - Cauchy if only if for
3. Any set is -closed if for every which is -convergent to implies .
4. Any set is -complete if for every -Cauchy sequence in is -convergent sequence in .
5. Any set is -bounded if the diameter of is bounded, that is

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1. The set satisfies -conditions if only if for any -convergent for some implies -convergent for all .

Note that the existence of the -limits of any which is -convergent is unique and the -convergent sequence is not always - Cauchy sequence. Thus, we define the -condition in generalized modular metric space [6]. This -condition implies convergent sequence in generalized modular metric space is Cauchy sequence.

*Definition 2.3* Let be a generalized modular metric space and . A mapping , where is nonempty, is called -contraction mapping if there is such that

for all .

Here is the result of the fixed point theorem for contraction mapping in generalized modular metric space. The concept of the proof is similar to the proof of the Banach Contraction Principle.

*Theorem 2.4* [14] Let be an arbitrary -complete generalized modular metric for some . If is a -contraction mapping on , for some nonempty, -closed, and -bounded then has aunique fixed point.

1. **Normal structure and compactness in generalized modular metric space**

The concept of normal structure in normed space was initiated by Gulevich in 1996. This concept is used for the fixed point theorem of nonexpansive mapping [5]. Now we start by defining convexity structure in generalized modular metric space which is motivated by some concepts in metric space. First, we define ball in generalized modular metric space.

*Definition 3.1* Let be an arbitrary generalized modular metric space and . For any and , the set is called the -ball which has a center in and radius .

For the rest of our discussion, we assume that the constant so the -ball is closed and we also assume that for all , so the -ball is always nonempty since the ball always contains its center.

*Definition 3.2.*Let be an arbitrary generalized modular metric space and . A set is said -admissible if can be stated as an intersection of -balls. Moreover, the collection of all admissible sets is denoted by .

Note that any nonempty set is not always can be stated as an intersection of -balls. Thus, we define the smallest admissible set which contains , i.e.

Any -bounded , we define , where . It is easy to see that .

*Definition 3.3* Let be an arbitrary generalized modular metric space, , and nonempty. The collection is said to have -normal structure if for every which is not a singleton and -bounded satisfies . Moreover, is said to have - uniformly normal structure if there is a constant such that for all which is not a singleton.

From Definition above, we can conclude that uniformly normal structure implies a normal structure.

*Definition* *3.4* Let be an arbitrary generalized modular metric space, , and nonempty. The collection is said -compact if only if for any where for any finite collection implies .

*Definition* *3.5* Let be an arbitrary generalized modular metric space, , and nonempty. The collection is said to have -property if for any sequence where and is nonempty for all implies

*Definition* 3.6 Let be an arbitrary generalized modular metric space, , and nonempty. A mapping is called -nonexpansive mapping if only if .

Here some properties that can be proved and useful for non-expansive mapping fixed point theorem.

*Properties 3.7* Let be an arbitrary generalized modular metric space and . If -compact then -complete.

*Lemma* 3.8 Let be an arbitrary generalized modular metric space, , and nonempty, -closed, -bounded, and -nonexpansive mapping. If -normal then there is such that and

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*Proof*. Let . **Case 1.**  Take and it completes the proof. **Case 2. .** Since -normal then . We define a set , then . Let . The set is nonempty since . Take , then , , and . Let then .

Note that and . This implies . Thus, , so that . Based on the definition of , we get . Thus, .

Let , then so . We get . Let , then . Since is -nonexpansive mapping, then . Moreover, for any , we get . Since then , it is equivalent to . Therefore, we get . It implies . Thus, . By definition of , we have . It concludes that .

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*Lemma* 3.9 Let be an arbitrary generalized modular metric space, , and nonempty, -closed, and -bounded. If -uniformly normal then has -property.

*Proof.* Let be an arbitrary sequence of nonempty sets where for all , and . Since -uniformly normal then there is such that .

Then for all , we define

The set and nonempty since for all . Then we define

Note that for all if then and for some . In this case, we can assume . Since then It implies . Repeat that construction, we can get a collection where for all , for every . Moreover, for arbitrary we get for all .

From that collection, we take Since then for . Thus, there is exist -Cauchy where for all . Since -complete then there is existed such that for . Since for all , -closed then for all . Therefore, we get .

1. **Fixed point theorems in generalized modular metric space**

The assumption in Theorem 2.4 can be weakened by (the composition of ) is a contraction for some . Here is the result of contraction mapping in generalized modular metric space by weakening the assumption.

*Theorem 4.1*Let be -complete generalized modular metric space for some . If - contraction mapping, for some non-empty, closed, -bounded then has a unique fixed point.

*Proof***.** From Theorem 2.4, we get has a unique fixed point, let is the fixed point of . Since then is also a fixed point of . Because of the uniqueness of the fixed point existence for then .

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For the nonexpansive mapping, we use the normal structure property.

*Theorem* 4.2Let be generalized modular metric space, , nonempty set which is -closed and - bounded. If - normal and compact, is - non-expansive mapping, then has a fixed point.

*Proof***.** Let Note that since .

Then we define for arbitrary set , . Let , then there is exist where such that . By continuing this process, we get for every there is where such that . Therefore, we have . Note that if we take , since compact and it can be shown that . By Lemma 3.8, then we get with property

Note that Thus by definition of we get for all , such that . It implies . By the assumption - normal, then is a singleton*,* let say , where . Thus, has a fixed point.

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*Theorem 4.3* Let , , nonempty, -bounded, and -closed. If - uniformly normal and - nonexpansive mapping, then has a fixed point

*Proof***.** Let then non empty, since . We define

Take and where such that

Next, for all we can choose where and Let then since satisfies -property. Moreover, it can be shown that . By Lemma 3.8, there is exist such that

Note that By using the definition of , we get, for all . Thus, . It implies By the asumption - uniformly normal, then is a singleton and the element of is a fixed point of .

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1. **Conclusions**

From the discussion above, we can conclude that if is a complete modular metric space and is a contraction mapping then has a unique fixed point. Moreover, we need the constant of generalized modular metric’s axiom should be less than or equal to 1 to guarantee the ball is closed. The closeness property of the ball is necessary to generate normal structure in generalized modular metric space. This normal structure property assures the existence of the fixed point of non-expansive mapping.

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